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INFLATION OF NONLINEARLY DEFORMED ANNULAR ELASTIC MEMBRANES

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Abstract—The inflation of an isotropic, nonlinear elastic membrane by means of a volume independent hydrostatic pressure is studied. In particular, we consider annular membranes which, on the inner boundary, have been subjected to an axial twist and a displacement normal to the plane of the membrane, with the outer boundary fixed. A direct two-dimensional approach is adopted and by introducing a relaxed strain-energy function the occurrence of wrinkled solutions is considered. The governing equations are found to reduce to a system of first order differential equations, which are then solved numerically for a Mooney–Rivlin material, over a range of boundary value problems.

1. INTRODUCTION

In a recent paper Roxburgh *et al.* (1995) considered the problem of determining the nonlinear elastic deformation of an isotropic annular membrane, which had been subjected to a transverse displacement and axial twist at the inner boundary, with the outer boundary being fixed. In general, whenever twisting phenomena are introduced the membrane surface will become wrinkled, see for example the paper by Li and Steigmann (1993) which considered the problem in the absence of a transverse displacement. These wrinkled solutions correspond to regions of compressive stress, for which the standard membrane strain-energy functions cannot be applied. This problem was addressed most recently by Pipkin (1986a) and Steigmann (1990) who showed that by introducing a 'relaxed' form of the strain-energy function, in effect the quasiconvexification of the strain-energy function, the resulting deformation minimizes the energy functional and thus can furnish the equilibrium solution. In the above papers by Roxburgh *et al.* (1995) and Li and Steigmann (1993) the effect of pressure was ignored and the aim of this paper is to extend these results by allowing for the inclusion of a volume-independent hydrostatic pressure. Such a problem could arise in the suspension of an automobile, where a rubber seal is used to enclose lubricating fluid between an inner strut and an outer jacket, which can slip and twist relative to one another.

Here we closely follow the approach adopted in the previous paper by Roxburgh *et al.* (1995) and note that while a brief review of the results contained therein is given, we refer back to this paper for further details. We take the standard direct two-dimensional approach to membrane theory, as proposed by Green *et al.* (1965) and Steigmann (1990), and consider the membrane to be initially planar. The membrane is then subjected to a radial prestretch and the inner boundary twisted and transversely displaced, while the outer boundary is kept fixed. This twisting means that the principal stretch directions can no longer be derived from simple geometry and that their values cannot be expressed in a compact form. A hydrostatic pressure p is then introduced and the equations of equilibrium are derived. Whenever a pressure is considered in problems of this type, there may be two or more solutions which satisfy the given boundary value problem; see for example Beatty (1987), Klingbeil and Shield (1964) and Khayat *et al.* (1992). It is found in general, that one of the possible solutions has principal stretches which are considerably smaller than the other solutions, if they occur. It is this solution with the smaller strains which would first occur on a path of pressure loading from the pressure free case, and so is the solution

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of primary interest. The larger strain solutions can only be reached on a path of pressure unloading after the pressure has been initially increased up to some critical value. However the principal stretches introduced by such a process are so large that they cannot be modelled accurately by the generally accepted constitutive relations used in the current literature; see Ogden (1984) for a discussion on the domains of confidence for several commonly used forms. We therefore restrict our attention to studying the primary inflation curve. Klingbeil and Shield (1964) circumvented this uniqueness problem by taking the radial principal stretch at the pole to parameterize the deformations instead of the pressure. The authors did not attempt to satisfy fixed boundary conditions at the outer edge of the disk, as opposed to the case here, and as such a similar approach would not be applicable here.

The approach adopted here is to consider a quasi-static pressure loading from the deformed state with no pressure present, for a given boundary value problem. In order to obtain explicit results we consider a Mooney–Rivlin material, which is known to hold only for maximum stretches of around 3. In this significant range the uniqueness problems outlined above are found not to occur. The numerical method used to solve the governing equations could easily be adapted for other strain-energy functions, and the full behaviour of all the possible solution branches investigated. However as can be seen from papers such as Klingbeil and Shield (1964) and Fulton and Simmonds (1986) appropriate choices of strain-energy function for membranes involving large stretches are not obvious.

For the Mooney–Rivlin material a range of problems are considered and solved numerically by means of a finite difference scheme, after introducing a suitable non-dimensional form for the governing equations. The results obtained are then displayed graphically and discussed in Section 4.

2. BASIC EQUATIONS AND NOTATION

We employ the direct two-dimensional approach proposed by Green *et al.* (1965) to describe the deformation of a nonlinear, isotropic hyperelastic membrane. The membrane is taken to occupy the region G_0 in a two-dimensional Euclidean space, in its reference configuration, which here is taken to be the undeformed configuration. Convected Gauss coordinates θ^α ($\alpha = 1, 2$) are chosen, so that a material point in G_0 with position vector $\mathbf{R}(\theta^1, \theta^2)$, will under a given deformation be mapped to the position $\mathbf{r}(\theta^1, \theta^2)$ lying on the deformed surface G of a three-dimensional Euclidean space. This induces a natural basis in the deformed configuration G

$$\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}, \quad \alpha = 1, 2, \tag{1}$$

where $(\ )_{,\alpha} = \partial(\ )/\partial\theta^\alpha$, which span the tangent plane of the deformed surface at $\mathbf{r}$, provided $\mathbf{r}_{,1} \times \mathbf{r}_{,2} \neq \mathbf{0}$. We note that in the following the summation convention is assumed, unless otherwise stated, and that Greek indices may take the values $\{1, 2\}$ while Latin indices may take the values $\{1, 2, 3\}$. The metric tensor corresponding to the basis in eqn (1) has components

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta, \tag{2}$$

while a unit normal to the deformed surface at $\mathbf{r}$ is

$$\mathbf{a}_3 = \frac{\mathbf{a}_1 \times \mathbf{a}_2}{a^{1/2}}, \tag{3}$$

where $a = \det\,(a_{\alpha\beta})$. Provided that $a > 0$, we can uniquely determine the dual basis by

$$\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta^\alpha_\beta, \quad a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta, \quad \mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta, \tag{4}$$

where δ^α_β is the Kronecker delta. Similarly, on the undeformed surface G_0, we can define

$$\mathbf{A}_\alpha = \mathbf{R}_{,\alpha}, \quad A_{\alpha\beta} = \mathbf{A}_\alpha \cdot \mathbf{A}_\beta, \quad A = \det(A_{\alpha\beta}) > 0,$$

$$\mathbf{A}_3 = \frac{\mathbf{A}_1 \times \mathbf{A}_2}{A^{1/2}}, \quad \mathbf{A}^\alpha \cdot \mathbf{A}_\beta = \delta^\alpha_\beta, \quad A^{\alpha\beta} = \mathbf{A}^\alpha \cdot \mathbf{A}^\beta. \tag{5}$$

Following Steigmann (1990), the deformation gradient $\mathscr{F}$ may be written as

$$\mathscr{F} = \mathbf{a}_\alpha \otimes \mathbf{A}^\alpha, \tag{6}$$

where $\otimes$ denotes the tensor, or dyadic, product. The Cauchy–Green stretch tensor $\mathscr{C} = \mathscr{F}^{\mathrm{T}}\mathscr{F}$, where $\mathscr{F}^{\mathrm{T}}$ denotes the transpose of $\mathscr{F}$, is then

$$\mathscr{C} = a_{\alpha\beta}\mathbf{A}^\alpha \otimes \mathbf{A}^\beta. \tag{7}$$

For hyperelastic materials having a strain-energy function $\mathscr{W}(\mathscr{F})$ per unit area of the reference surface, the Piola–Kirchhoff stress tensor $\mathscr{T}$ can be written in the form

$$\mathscr{T} = \mathbf{T}^\alpha \otimes \mathbf{A}_\alpha, \tag{8}$$

where $\mathbf{T}^\alpha$ are the resultant stress vectors. By objectivity, or frame-indifference, of the strain-energy function, that is $\mathscr{W}(\mathscr{F}) = \mathscr{W}(\mathscr{Q}\mathscr{F})$ for all proper orthogonal $\mathscr{Q}$, we may write

$$\mathscr{W}(\mathscr{F}) = \hat{\mathscr{W}}(\mathscr{C}), \tag{9}$$

so that the resultant stress vectors $\mathbf{T}^\alpha$ have the form

$$\mathbf{T}^\alpha = 2\frac{\partial \hat{\mathscr{W}}}{\partial a_{\alpha\beta}}\mathbf{a}_\beta. \tag{10}$$

In the presence of a hydrostatic pressure p, the equilibrium equations have the form, see Steigmann (1990),

$$A^{-1/2}\frac{\partial}{\partial\theta^\alpha}(A^{1/2}\mathbf{T}^\alpha) + pJ\mathbf{a}_3 = \mathbf{0}, \quad \theta^\alpha \in G_0 \tag{11}$$

where $J = (\det\mathscr{C})^{1/2} = (a/A)^{1/2}$.

If we let λ_1^2, λ_2^2 denote the eigenvalues of $\mathscr{C}$, then we refer to λ_1 and λ_2 as the principal stretches of the deformation. If the membrane is isotropic, eqn (9) then implies that

$$\mathscr{W}(\mathscr{F}) = \tilde{\mathscr{W}}(\lambda_1, \lambda_2), \tag{12}$$

where $\tilde{\mathscr{W}}$ is a symmetric function of its arguments. The principal Biot stresses t_α are then defined as

$$t_\alpha = \frac{\partial\tilde{\mathscr{W}}}{\partial\lambda_\alpha}, \quad \alpha = 1, 2. \tag{13}$$

For the problem considered here, it is expected that the membrane may become wrinkled in certain regions, corresponding to points where $t_\alpha < 0$. As was shown by Pipkin

(1986a) necessary conditions for any deformation to be a minimizer of the total strain-energy over the membrane are

$$t_\alpha \geqslant 0, \quad \alpha = 1, 2 \tag{14}$$

at all points, which therefore excludes any deformation which induces compressive stresses at any point. The inequaltities (14) are also necessary for the two-dimensional Legendre–Hadamard conditions, which in turn are necessary for the deformation to be *infinitesimally stable*, in the sense of Truesdell and Noll (1965).

In an attempt to ensure that the inequalities (14) are satisfied for any deformation Pipkin studied continuously distributed wrinkles and introduced (Pipkin 1986b) a relaxed strain-energy function $\mathscr{W}_r$, which is derived from $\tilde{\mathscr{W}}$ in eqn (12) in the following manner.

Suppose that an initially unstressed membrane is subjected to simple tension in the λ_1 direction, then the *natural width* in simple tension is defined to be the implicit solution for λ_2 from the equation

$$t_2 = \frac{\partial \tilde{\mathscr{W}}}{\partial \lambda_2}(\lambda_1, \lambda_2) = 0 \tag{15}$$

which we write as $\lambda_2 = \omega(\lambda_1)$. We assume that ω is uniquely determined from eqn (15) and that it is a smooth function of its argument. This means that if $\lambda_2 < \omega(\lambda_1)$ for any pair (λ_1, λ_2) then the definition eqn (13) gives that $t_\alpha < 0$.

The relaxed strain-energy function is derived from eqn (12) by replacing λ_2 by $\omega(\lambda_1)$ whenever $\lambda_2 \leqslant \omega(\lambda_1)$, or vice versa if $\lambda_1 \leqslant \omega(\lambda_2)$. This leads to the explicit definition

$$\mathscr{W}_r(\lambda_1, \lambda_2) = \begin{cases} 0, & 0 < (\lambda_1, \lambda_2) \leqslant 1, \\ \tilde{\mathscr{W}}(\lambda_1, \omega(\lambda_1)), & \lambda_1 > 1, \quad \lambda_2 \leqslant \omega(\lambda_1), \\ \tilde{\mathscr{W}}(\omega(\lambda_2), \lambda_2), & \lambda_1 \leqslant \omega(\lambda_2), \quad \lambda_2 > 1, \\ \tilde{\mathscr{W}}(\lambda_1, \lambda_2), & \lambda_1 > \omega(\lambda_2), \quad \lambda_2 > \omega(\lambda_1), \end{cases} \tag{16}$$

from which it can be shown that $\mathscr{W}_r$ is the quasiconvex form of $\tilde{\mathscr{W}}$. On taking the principal Biot stresses with respect to $\mathscr{W}_r$, that is $t_\alpha = \partial \mathscr{W}_r / \partial \lambda_\alpha$, then it is easy to show that the inequalities in eqn (14) are automatically satisfied everywhere. Pipkin (1986b) showed that the relaxed strain-energy function [eqn (16)] also automatically satisfies several other conditions necessary for the Legendre–Hadamard condition. For "incompressible" elastic membranes, which are taken to behave as for an incompressible three-dimensional material restricted to two-dimensions, the natural width is given by

$$\omega(\lambda_\alpha) = \lambda_\alpha^{-1/2}. \tag{17}$$

In the following sections we will be interested in finding the relationship between the pressure and the volume under the deformed surface of the membrane. In terms of the position vector $\mathbf{r}$, the volume $V[\mathbf{r}]$ lying between the membrane surface and some chosen lower surface is given by, Steigmann (1991),

$$\begin{aligned} V[\mathbf{r}] &= \frac{1}{3}\int_{G_0} \mathbf{r}\cdot(\mathbf{r}_{,1} \times \mathbf{r}_{,2}) A^{-1/2}\, \mathrm{d}S \\ &= \frac{1}{3}\int_{G_0} J\mathbf{r}\cdot\mathbf{a}_3\, \mathrm{d}S, \end{aligned} \tag{18}$$

where $\mathrm{d}S = A^{1/2}\,\mathrm{d}\theta^1\,\mathrm{d}\theta^2$ is the elemental area on the reference surface G_0.

3. GOVERNING EQUATIONS

We consider the same boundary value problem as in Roxburgh *et al.* (1995), but this time we also include the effects of a hydrostatic pressure p. As in this previous paper, in the reference configuration we introduce cylindrical polar coordinates (R, Θ, Z), with corresponding orthonormal basis $(\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_Z)$, so that the membrane initially occupies the annular region

$$\mathbf{R} = R\mathbf{e}_R, \tag{19}$$

with $A \leqslant R \leqslant B$, $0 \leqslant \Theta \leqslant 2\pi$, $Z = 0$. The membrane is subjected to a radial stretch

$$\mathbf{r} = \lambda_0 \mathbf{R}, \tag{20}$$

so that it occupies the region with $a = \lambda_0 A \leqslant r = \lambda_0 R \leqslant \lambda_0 B = b$, and in general we assume that $\lambda_0 \geqslant 1$. The inner rim, $r = a$, is then displaced an amount w_0 in the positive Z-direction, and twisted through a positive angle φ_0, while the outer rim $r = b$ is kept fixed. With the assumption that the resulting deformation is axisymmetric, we take the deformation to be of the form

$$\mathbf{r} = u(R)\cos\varphi(R)\mathbf{e}_R + u(R)\sin\varphi(R)\mathbf{e}_\Theta + w(R)\mathbf{e}_Z, \tag{21}$$

where $u(R)$ is the deformed radius, $w(R)$ is the transverse displacement and $\varphi(R)$ is the angle of twist of the material circle with undeformed radius R. The boundary conditions are thus

$$\begin{aligned} u(A) &= \lambda_0 A, \quad & u(B) &= \lambda_0 B, \\ w(A) &= w_0, \quad & w(B) &= 0, \\ \varphi(A) &= \varphi_0, \quad & \varphi(B) &= 0. \end{aligned} \tag{22}$$

If we take the Gauss coordinates as

$$\theta^1 = R, \text{and } \theta^2 = \Theta,$$

it follows, on differentiating eqns (19) and (21) that

$$\begin{aligned} \mathbf{A}_1 &= \mathbf{A}^1 = \mathbf{e}_R, \quad \mathbf{A}_2 = R\mathbf{e}_\Theta = R^2\mathbf{A}^2, \\ \mathbf{a}_1 &= (u\cos\varphi)'\mathbf{e}_R + (u\sin\varphi)'\mathbf{e}_\Theta + w'\mathbf{e}_Z, \\ \mathbf{a}_2 &= -u\sin\varphi\mathbf{e}_R + u\cos\varphi\mathbf{e}_\Theta, \end{aligned} \tag{23}$$

where $\mathrm{d}(\ )/\mathrm{d}R = (\ )'$. On substituting from eqn (23) into eqns (2) and (5) the corresponding metric tensors can be derived and yield

$$a = \det(a_{\alpha\beta}) = u^2(u'^2 + w'^2), \quad A = \det(A_{\alpha\beta}) = R^2, \tag{24}$$

which allows us to find the unit normal $\mathbf{a}_3$, from eqn (3),

$$\mathbf{a}_3 = \frac{-(w'\cos\varphi\mathbf{e}_R + w'\sin\varphi\mathbf{e}_\Theta - u'\mathbf{e}_Z)}{(u'^2 + w'^2)^{1/2}}. \tag{25}$$

Equation (7) gives that the Cauchy–Green stress tensor $\mathscr{C}$ has the form

$$\mathscr{C} = (u'^2+w'^2+u^2\varphi'^2)\mathbf{e}_R\otimes\mathbf{e}_R+\frac{u^2\varphi'}{R}(\mathbf{e}_R\otimes\mathbf{e}_\Theta+\mathbf{e}_\Theta\otimes\mathbf{e}_R)+\frac{u^2}{R^2}\mathbf{e}_\Theta\otimes\mathbf{e}_\Theta, \tag{26}$$

which has eigenvalues satisfying

$$\det\mathscr{C} = \lambda_1^2\lambda_2^2 = \frac{u^2}{R^2}(u'^2+w'^2),$$

$$\operatorname{tr}\mathscr{C} = \lambda_1^2+\lambda_2^2 = u'^2+w'^2+\frac{u^2}{R^2}(R^2\varphi'^2+1). \tag{27}$$

Note that when no twisting is present, that is $\varphi' = 0$, then eqn (27) immediately gives $\lambda_1^2 = u^2/R^2$ and $\lambda_2^2 = u'^2+w'^2$, but in the general case no such simplification can be made. In order to obtain a more manageable form for the governing equations, we set

$$\Omega = (u'^2+w'^2)^{1/2}, \quad \alpha = \cot^{-1}\left(\frac{u'}{w'}\right), \quad \psi = \frac{u}{R}, \tag{28}$$

so that

$$u' = \Omega\cos\alpha, \quad w' = \Omega\sin\alpha, \tag{29}$$

and $\alpha(R)$ is thus the angle a tangent line to the meridian, at a material radius R, makes with the outward radial axis. With this, eqn (27) simplifies to

$$J = (a/A)^{1/2} = \lambda_1\lambda_2 = \Omega\psi, \quad \lambda_1^2+\lambda_2^2 = \Omega^2+\psi^2(1+R^2\varphi'^2). \tag{30}$$

The resultant stress vectors given by eqn (10) become, after some calculation

$$\mathbf{T}^1 = (D-\psi^2C)\mathbf{a}_1+(\psi^2\varphi'C)\mathbf{a}_2,$$

$$\mathbf{T}^2 = (\psi^2\varphi'C)\mathbf{a}_1+\frac{1}{R^2}\{D-(\Omega^2+\psi^2R^2\varphi'^2)C\}\mathbf{a}_2, \tag{31}$$

where the constitutive dependence of eqns (31), enters solely through the terms

$$C = \frac{t_1/\lambda_1-t_2/\lambda_2}{\lambda_1^2-\lambda_2^2}, \quad D = \frac{\lambda_1 t_1-\lambda_2 t_2}{\lambda_1^2-\lambda_2^2}, \tag{32}$$

where the $t_\alpha = \partial\tilde{W}/\partial\lambda_\alpha$ are the Biot principal stresses.

In the presence of a hydrostatic pressure p, the equilibrium equation (11) becomes

$$\frac{1}{R}\frac{\partial}{\partial R}(R\mathbf{T}^1)+\frac{1}{R}\frac{\partial}{\partial\Theta}(R\mathbf{T}^2)+p\psi\Omega\mathbf{a}_3 = \mathbf{0}, \tag{33}$$

where the $\mathbf{T}^\alpha$ are given by eqn (31) and $\mathbf{a}_3$ is given by eqn (25).

On performing the differentiation in eqn (33) and separating into the $\mathbf{e}_R$, $\mathbf{e}_\Theta$ and $\mathbf{e}_Z$ components, eqn (33) yields three equations of equilibrium. Elimination of the trigonometric terms in φ between the components in the $\mathbf{e}_R$ and $\mathbf{e}_\Theta$ directions gives

$$[u^2R\varphi'D]' = 0 \tag{34}$$

and

$$[R(D-\psi^2 C)u']' = \psi\{D(1+R^2\varphi'^2)-\Omega^2 C\}+puw', \tag{35}$$

while the component in the $\mathbf{e}_Z$ direction is

$$[R(D-\psi^2 C)w']'+\frac{p}{2}(u^2)' = 0, \tag{36}$$

which can be integrated immediately when p is independent of the deformation, that is $\mathrm{d}p/\mathrm{d}R = 0$. By eliminating p between eqn (35) and (36) and integrating eqn (34), we obtain the equilibrium equations

$$\begin{aligned} Ru^2\varphi' D &= M, \\ R(D-\psi^2 C)w'+\frac{p}{2}u^2 &= N, \\ [R(D-\psi^2 C)\Omega]' &= \psi\{(1+R^2\varphi'^2)D-\Omega^2 C\}\cos\alpha, \end{aligned} \tag{37}$$

with M and N suitable constants, where N can be interpreted as the transverse pull-out force and M the torque, applied at the inner boundary. Alternatively, on combining with eqn $(37)_2$ then eqn $(37)_3$ can be rewritten as

$$\alpha' = -\frac{\psi\{(1+R^2\varphi'^2)D-\Omega^2 C\}\sin\alpha+pu\Omega}{R(D-\psi^2 C)\Omega}. \tag{38}$$

We now consider how these equations are modified when the solution lies within a wrinkled region. As was noted in section 2, wrinkled regions occur whenever $t_\alpha = \partial\tilde{\mathscr{W}}/\partial\lambda_\alpha < 0$ $(\alpha = 1, 2)$, but in this present case only one of these t_α can ever become negative and we label this stress as t_2. We now replace the standard strain-energy $\tilde{\mathscr{W}}$ by its relaxed form $\mathscr{W}_r$ given by eqn (16), with $\lambda_2 = \omega(\lambda_1)$. This means that $t_2 = \partial\mathscr{W}_r/\partial\lambda_2 \equiv 0$, whenever $\partial\tilde{\mathscr{W}}/\partial\lambda_2 \leqslant 0$. The constitutive terms C and D from eqn (32) in a wrinkled region thus simplify to

$$C = \frac{t_1(\lambda_1, \omega(\lambda_1))}{\lambda_1(\lambda_1^2-\lambda_2^2)}, \quad D = \lambda_1^2 C. \tag{39}$$

Note that we only replace λ_2 by $\omega(\lambda_1)$ within the terms directly involving the strain-energy function. It follows that in a wrinkled region the equilibrium equations are again given by eqn (37), but with eqn (32) replaced by eqn (39).

Finally in this section, for the deformation given by eqn (21) the volume $V[\mathbf{r}]$ contained between the deformed membrane surface and the $Z = 0$ plane can be shown from eqns (18), (25) and (30) to be

$$V[\mathbf{r}] = \frac{2\pi}{3}\int_A^B u(u'w-uw')\,\mathrm{d}R. \tag{40}$$

4. NUMERICAL METHOD AND RESULTS

In order to solve the governing equations numerically, we introduce the non-dimensionalized variables

$$\hat{R} = \frac{R}{B}, \quad \hat{A} = \frac{A}{B}, \quad \hat{B} = 1, \quad \hat{u} = \frac{u}{B}, \quad \hat{w} = \frac{w}{B},$$

$$\hat{\mathscr{W}} = \frac{\mathscr{W}}{\mu}, \quad \hat{t}_1 = \frac{t_1}{\mu}, \quad \hat{t}_2 = \frac{t_2}{\mu}, \quad \hat{p} = \frac{pB}{\mu}, \tag{41}$$

where μ is a material constant, related to the shear modulus, with dimensions of energy per unit area. The hat notation is subsequently dropped and all quantities are taken to be in their nondimensional form. These quantities can then be substituted into eqns (28)–(30), (37) and (38), together with either eqn (32) or (39) as appropriate, to yield a system of first order ordinary differential equations to be solved. In order to solve this system explicitly, a particular form for the strain-energy function must be chosen. Here we consider an "incompressible" Mooney–Rivlin material, with nondimensional form

$$\tilde{\mathscr{W}}(\lambda_1, \lambda_2) = \tfrac{1}{2}\{\gamma(\lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} - 3) + (1-\gamma)(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2\lambda_2^2 - 3)\}, \tag{42}$$

where γ, $0 < \gamma \leqslant 1$ is a constant; for all the particular results considered in this section the value $\gamma = 0.9$ was chosen. As was noted previously there may exist several solutions for a given pressure p, however it is only the small strain solution corresponding to the primary inflation curve that we are interested in here. To ensure that this particular solution is found, any problem is first solved in the absence of pressure and then repeatedly solved as the pressure is incremented from zero. At each stage the volume enclosed under the membrane is calculated from eqn (40) as a further verification that no jump in solution branch occurs.

A shooting method was chosen to solve the governing system of ordinary differential equations. This approach involves choosing an initial guess for the quantities λ_1, λ_2 and α at the inner rim $R = A$. These parameters are then adjusted according to the values of u, w and φ obtained at the outer rim $R = 1$, as compared with the required boundary conditions given in eqn (22). This adjustment must be carried out manually as no definitive method was found to do this automatically. These parameter values are thus continually adjusted until the boundary conditions in eqn (22) are satisfied.

By considering eqn (30), it can be seen that for any solution, we must have

$$\lambda_1(R) > \lambda_0 \geqslant \lambda_2(R), \quad A \leqslant R \leqslant 1 \tag{43}$$

and since, in general, the prestretch $\lambda_0 \geqslant 1$, the condition for wrinkled solutions to occur is $\lambda_2 < \omega(\lambda_1)$, which for an incompressible material corresponds to

$$\lambda_1\lambda_2^2 < 1. \tag{44}$$

Figure 1 displays the typical behaviour of an initially undeformed membrane which is subject to increasing pressure. Figure 1(a) shows the radial profiles of the deformed surface for a range of pressures, while Fig. 1(b) displays the pressure-volume relationship. Note that as the pressure is increased beyond $p = 3$ the resulting volume increase is much greater. This increase in volume involves a correspondingly large increase in the principal stretches induced and even when $p = 4$ the stretch λ_1 is significantly greater than 3 near the inner boundary. As the pressure is then increased beyond $p = 4$ the resulting value of λ_1 near $R = A$ is found to increase rapidly beyond any realistic bounds.

In Fig. 2 a typical value for the pressure is chosen, namely $p = 2$, and the effects of both twisting and transversely displacing of the inner boundary of the membrane are displayed. From Figs 2(a) and 2(b) it is clear that while twisting effects do not greatly alter the profile of the deformed surface, they do have a considerable effect on the principal stretches induced. This is significant in that it is the principal stretches which determine whether wrinkling phenomena occur or not. From eqn (44) wrinkled solutions occur when $\lambda_1\lambda_2^2 < 1$ and in Fig. 2(c) the behaviour of the expression $\lambda_1\lambda_2^2$ over the membrane is

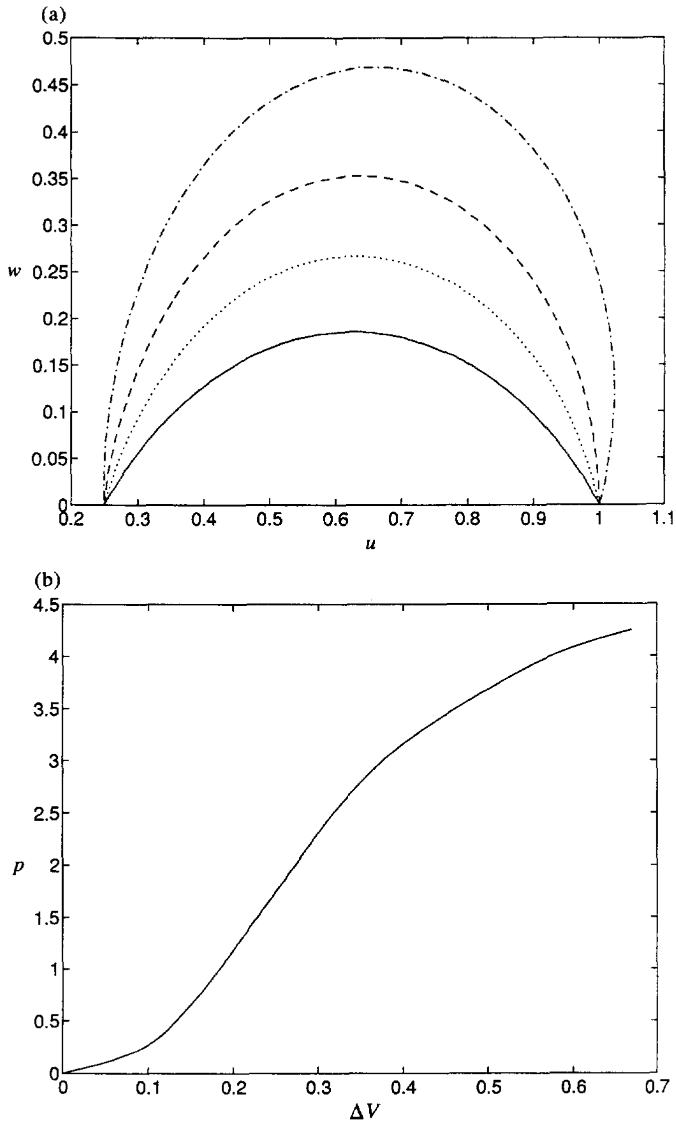

Fig. 1. Inflation of an initially undeformed flat Mooney–Rivlin membrane, with inner radius $A = 0.25$, is considered. Here (a) displays the profiles w against u, with the solid line corresponding to $p = 1$, the dotted line $p = 2$, the dashed line $p = 3$ and the chained dashed line to $p = 4$. In (b) the relationship between the pressure p and the increase in volume ΔV is displayed.

illustrated. Here, with $p = 2$, all four of the problems considered have $\lambda_1\lambda_2^2 > 1$ everywhere so that no wrinkled solutions arise. However, both of the deformations with twisting present contain wrinkled regions in the unpressurized state, see Roxburgh *et al.* (1995) for related examples, and as the value of p is increased these wrinkled regions are reduced in size as the corresponding principal stretches are increased. Thus as expected as the membrane is inflated any wrinkles are smoothed out by the increasing pressure.

Note that as is displayed by Fig. 2(b), the principal stretch λ_1 takes its largest value at the inner boundary and then drops rapidly in size as it approaches the outer edge. This is especially true when twisting effects are present and it is only exaggerated as the pressure is increased.

Finally in Fig. 3 we consider the effect that a prestretch has on the behaviour of the inflation of the membrane. The same problem is considered as in Fig. 2, but with a radial prestretch of $\lambda_0 = 1.25$ applied; in particular, Figs 3(a) and 3(b) may be compared with Figs 2(a) and 2(c), respectively. It is clear that the membrane deforms in a similar way in both cases, however when the prestretch is applied the principal stretches induced are larger.

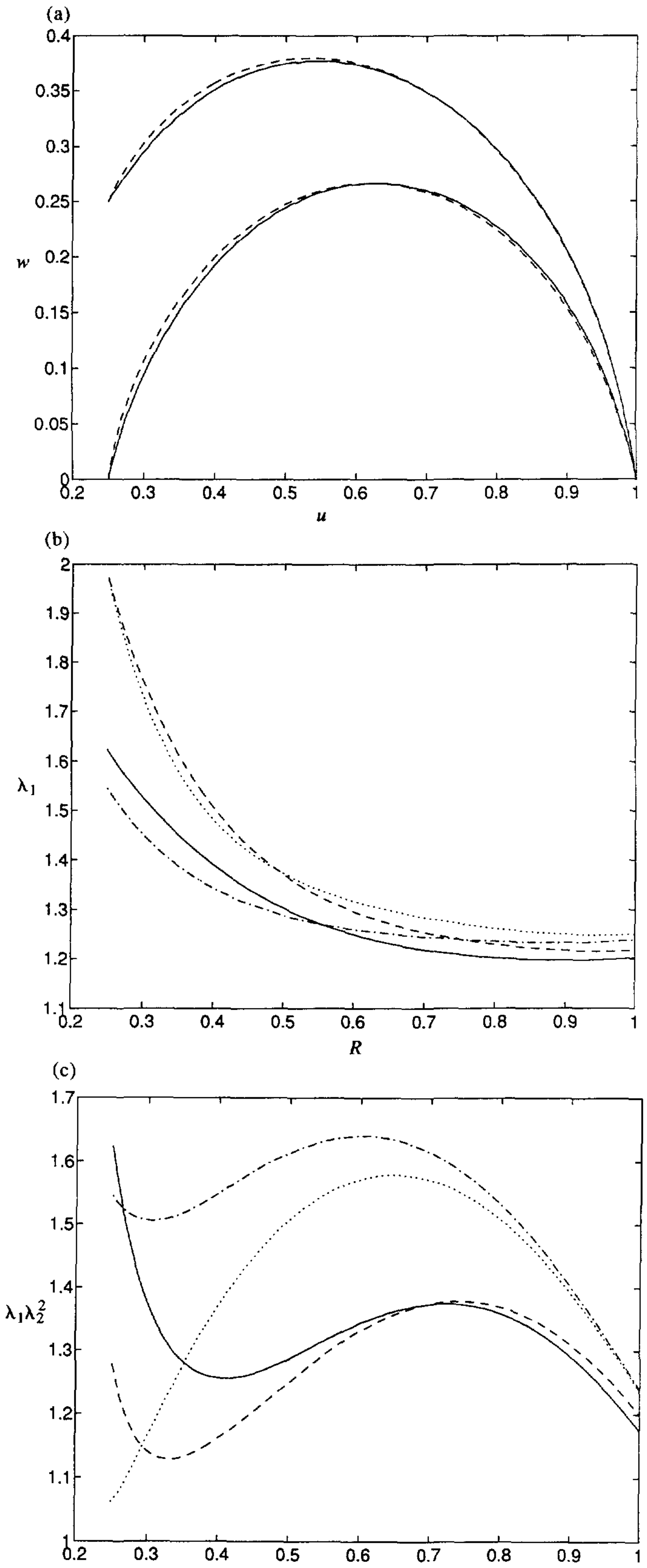

Fig. 2. For a given pressure $p = 2$ the behaviour of an initially unstretched Mooney–Rivlin membrane, with inner radius $A = 0.25$, is considered. In (a) the profiles w against u are displayed for $w_0 = 0$ and $w_0 = 0.25$, where the solid lines correspond to the situation with no twisting present, $\varphi_0 = 0$, and the dashed lines to the case with $\varphi_0 = 0.5$. In (b) λ_1 is displayed against R, while in (c) $\lambda_1\lambda_2^2$ is displayed against R, where in both cases the solid lines correspond to $w_0 = 0$, $\varphi_0 = 0$, the dotted lines to $w_0 = 0$, $\varphi_0 = 0.5$, the chained dashed lines to $w_0 = 0.25$, $\varphi_0 = 0$ and the dashed lines to $w_0 = 0.25$, $\varphi_0 = 0.5$.

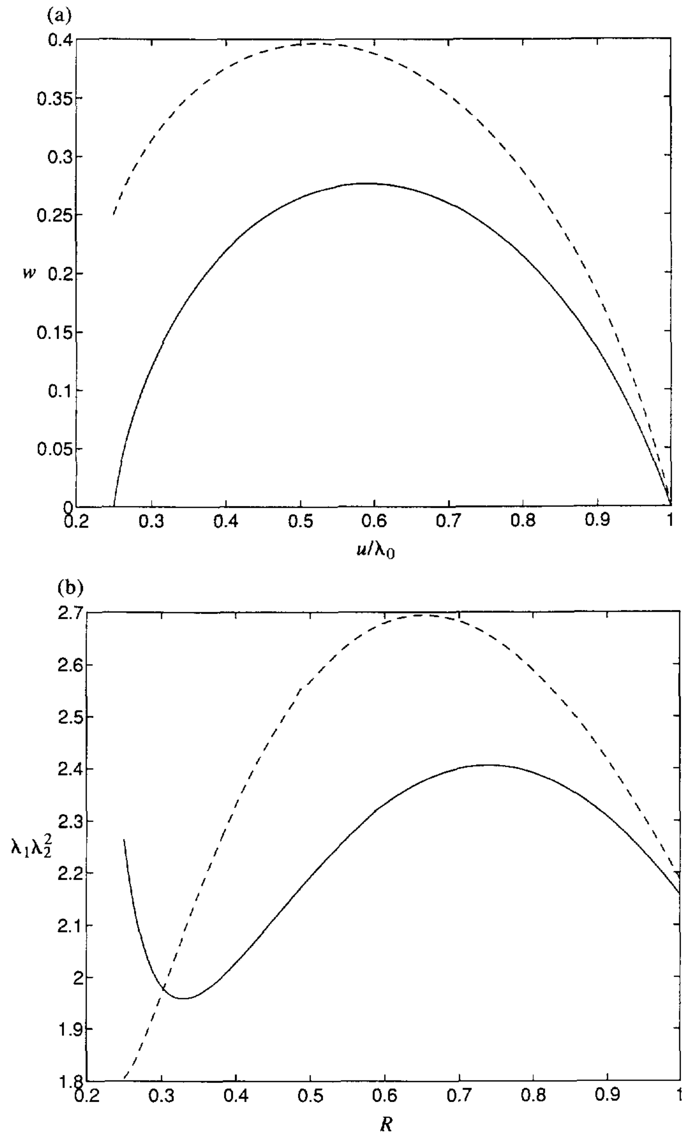

Fig. 3. For a Mooney–Rivlin membrane, with $A = 0.25$, subjected to a prestretch of $\lambda_0 = 1.25$, together with a pressure $p = 2$ and a twist $\varphi_0 = 0.5$, the profiles w against u/λ_0 are displayed in (a), while in (b) $\lambda_1\lambda_2^2$ is displayed against R. In both graphs the solid line corresponds to $w_0 = 0$ and the dashed line to $w_0 = 0.25$.

From the point of view of this paper, this means that a prestretch acts to inhibit the occurrence of wrinkling.

5. CONCLUSIONS

The object of this paper is to study a particular problem which necessarily gives rise to wrinkled solutions by using the formulation, involving relaxed strain-energy functions and continuously distributed wrinkles, proposed by Pipkin (1986a,b). It is found that the same numerical procedure can be used to solve for both wrinkled and tense regions, with only a simple check and change of two constitutive terms required to differentiate between both possibilities. The typical behaviour of solutions to this problem is displayed graphically and it is found that, as expected, the introduction of a pressure acts to smooth out any wrinkled regions that may occur, that is, as the pressure is increased the membrane becomes tense everywhere.

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